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OPTICAL SOLUTIONS FOR THE UNBOUNDED SUBSET-SUM PROBLEM

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Abstract. *A special computational device which uses light rays in order to decide whether there is a solution for the unbounded subset-sum problem is described in this paper. The device has a multigraph-like representation and the light traverses it following the routes given by the connections between the nodes. The graph has* 3 *nodes: the first one is where the light enters in the device; the second one is used for computing the solution (the computational node) and the* 3^{rd} *one is used for collecting the solution (the destination node). The computational node has a number of loops equal to the cardinal of the given set. To each loop (arc) we assign a number from the given set. The nodes are connected by an arc having been assigned a constant value. When the light passes through an arc it is delayed by the amount of time indicated by the number assigned to that arc. At the destination node we will check if there is a ray whose total delay is equal to the target value of the subset sum problem. In this way we have provided a solution to the YES/NO decision problem.*

Keywords: unconventional computing, optical computing, NP-complete, unbounded subset-sum problem

1. Introduction.

Optical computing refers to the use of light (instead of electricity) for manipulating, storing and transmitting data. It is hoped that in the near future optical computers will perform better than standard electronic computers.

Here we show how to solve a NP-complete problem (the unbounded subset-sum) by using an optical device. The problem is concerned with finding whether a given positive number B can be decomposed into a sum, having been given only the elements from a given set A. Each element of A is allowed to have any number of occurrences in the considered sum. Actually, we have to find if there is such a thing as a multiset composed of elements from A and having the sum B. The multiset is a variation of a set, but it accepts repeated values. Formally, a multiset maps unique elements to positive integers, indicating the multiplicity of that element.

The proposed device, which is very simple, has a multigraph-like structure. A multigraph or pseudograph is a graph which is allowed to have multiple edges, (i.e. edges that have the same end nodes). In other words, two nodes may be connected by more than one edge [18]. A multigraph also allows the usage of loops (i.e. edges with both ends on the same node).

The nodes are connected in such a way that all the possible multisets with elements from A can be generated. The multigraph has 3 nodes: one for initiating the signal (the start node), one for computing the solution (the computational node) and the 3^{rd} one is used for collecting the solution (the destination node). The computational node has $card(A)$ loops. To each loop, we assign a number from the given set A. The length of an arc is directly proportional to the number assigned to it. The computational node is also connected to the destination node by an arc to which we have assigned a constant value.

This research was motivated by the need of answering two important questions:

- Are there any other ways to perform computations alongside the existing ones?
- Can light be used for performing useful computations?

The paper is organized as follows: Related work in the field of optical computing for NP-complete problems is described in section 2 The unbounded subset-sum problem is described in section 3. The proposed device and the list of components required by this system are presented in section 4. The complexity and difficulties encountered by the proposed device are discussed in section 5. Suggestions for improving the device are given in section 6. Strength and weaknesses are discussed in section 7. Further work directions are given in section 8.

2. **Related optical devices for NP-complete problems.** Our purpose is to solve NP-complete problems. That is why we also review optical devices specially designed for tackling NP-complete problems. Currently, acording to our knowledge, 5 problems have been solved using optical principles similar to those presented in this paper. They are: the Hamiltonian Path problem [13, 14], the Traveling Salesman Problem [3, 17, 10], the standard subset sum problem [15, 19], the Exact Cover [16] and the Diophantine equations [12].

A system which solves the Hamiltonian path problem (HPP) [8] by using light and its properties has been proposed in [13, 14]. The device has the same structure as the graph where the solution is to be found. The light is delayed within the nodes, whereas the delays introduced by the arcs are constants. Because the problem requires that each node should be visited exactly once, a special delaying system was designed. At the destination node the ray which has visited each node exactly once is filtered out. This is very easy due to the special properties of the delaying system.

An optical device for the standard subset sum problem (each element can appear no more than once in the solution) was proposed in [15]. The basic idea was to generate all the possible subsets and then to select the good one (whose sum of numbers is equal to the target sum - denoted by B here). Please note that the generation of all the possible subsets is done in $O(B)$ time, but with an exponential consumption of energy.

In this paper we show how the unbounded version of the subset sum problem can be solved more easily, using fewer resources. The solution here requires only $n+1$ cables and 2 nodes (one of them containing a beam splitter). In [15] the solution for the standard subset sum requires $2*n$ cables and n beam-splitters.

An optical solution for solving the traveling salesman problem (TSP) was proposed in [17]. The power of optics in this method was shown by using a fast matrix-vector multiplication between a binary matrix, representing all the feasible TSP tours, and a gray-scale vector, representing the weights among the TSP cities. The multiplication was performed optically by using an optical correlator. In order to synthesize the initial binary matrix representing all the feasible tours, an efficient algorithm was provided. However, since the number of all the tours is exponential, the method is difficult to be implemented even for small instances.

TSP was also solved in [10] by using a similar idea as in [13, 14]. The identification of solutions was done by white light interferometry.

In the case of Exact Cover the original problem was decomposed in 2 subproblems: generating all the subsets of the given set and then selecting the correct one [16]. For the first step we have designed a light-based device which has a graph-like structure. Each arc can either represent an element of C or can be a skipping arc. An arc will actually delay the signal (light) which passes through by a certain amount of time. The nodes are connected by arcs in such a way that all the possible subsets of C are generated. In the second part we have assigned a special positive number to each item from the set to be covered, so that the sum of all the numbers assigned to that set is not equal to any other combination of numbers assigned to items from the set. These numbers have the same property as in the case of the optical solution for the Hamiltonian Path problem [13].

In [4] another brute-force approach was used for solving problems. Seven NP-complete problems have been solved in this way: Hamiltonian path, Traveling Salesman, Clique, Independent Set, Vertex Cover, Partition, 3-SAT, and 3D-matching.

An optical system which finds solutions to the 6-city TSP using a Kohonen-type network was proposed in [3]. The system shows robustness with regard to the light intensity fluctuations and weight discretization which have been simulated. The authors have suggested that a relatively large number of TSP cities could be handled by using this method.

3. **The unbounded subset-sum problem.** The description of the unbounded subset-sum problem [8, 9] is the following:

Given a set of real positive numbers $A = \{a_1, a_2, ..., a_n\}$ and a positive number B. Are there some integer values $x_i \geq 0$ $(1 \leq i \leq n)$ such that $\sum_{i=1}^{n}(x_i * a_i) = B$?

We are interested in the YES/NO decision problem. That is, we are interested to find whether a solution exists. We are not interested in finding the actual values of x_i. The problem belongs to the class of NP-complete problems [8]. No polynomial time algorithm is known to be a proper way of solving it.

However, a pseudo-polynomial time algorithm does exist for this problem [8]. The complexity of this algorithm is bounded by both n and B. The algorithm requires $O(B)$ storage space.

4. **The proposed device.** This section is devoted to an in-depth description of the proposed system. Section 4.1 describes the properties of light which are useful for our device. Section 4.2 introduces the operations performed by the components of our device. The basic idea behind our concept is given in section 4.3. The physical implementation of the labeling system is described in section 4.5.

4.1. **Useful properties of the light.** Our idea is based on two properties of light:

- The speed of light has a limit. The value of the limit is not very important at this stage of the explanation. The speed will become important when we compute the size of the instances that can be solved by our device (see section 5.4). What is important now is the fact that we can delay the ray by forcing it to pass through an optical fiber cable of a certain length.
- The ray can be easily divided into multiple rays of smaller intensity/power. Beam-splitters are used for this operation.

4.2. **Operations performed within our device.** The proposed device has a multigraph-like structure. Generally speaking, one operation is performed when a ray passes through a node and another one when it passes through an edge.

- When the ray passes through an arc it is delayed by the amount of time assigned to that arc. An arc is implemented by using an optical fiber [6] of a certain length.
- When the ray passes through a node it is divided into a number of rays equal to the external degree of that node. Every ray obtained is directed towards one of the nodes connected to the current node.

At the destination node we check if there is a ray which has arrived there at moment B (plus a constant value).

4.3. **Basic idea.** The idea for our device is that numbers from the given set A represent the delays induced to the signals (light) that pass through our device. For instance, if the numbers a_1, appearing twice, and a_4 generate the expected multiset, then the total delay of the signal should be $2 * a_1 + a_4$. By using light we can easily induce some delays by forcing the ray to pass through an optical cable of a given length.

This is why we have designed our device as a directed graph. The arcs which are implemented by using optical cables are labeled with numbers from the given set A. There is a total of 3 nodes. One of them is where the light is sent into the device. Another has $card(A)$ loops and is also connected by one arc to the 3^{rd} one, representing the destination node. A possible representation of this device is shown in Figure 1.

In the computational node we place a beam-splitter which will split a ray into $card(A) + 1$ subrays of smaller intensity.

The device will generate all possible multisets of elements from the given set A. Each combination will delay one of the rays by an amount of time equal to the sum of the lengths of the arcs in that path. It can be easily seen that each element of the set can have any number of occurrences.

We have assigned a positive constant value k to the arc that connects the computational node to destination. That is why, instead of waiting for a solution at moment B, we will have to wait for it at moment $B + k$, since all the multisets will have the k constant added.

Please note that this device is much simpler than the one for the standard subset sum. The number of optical cables in the current device was reduced to half when compared to the number of cables for the standard subset sum problem. Also, the current device requires 1 beam splitter while the device for the standard subset sum requires n beam splitters. However, here we have to deal with a complex beam-splitter which divides each ray into $n + 1$ subrays, whereas in the case of the standard subset sum we only have simple beam splitters which divide each ray into 2 subrays.

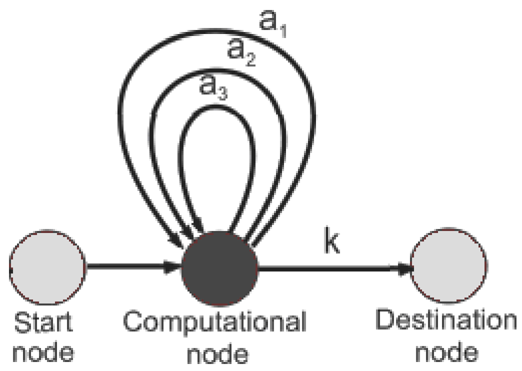

FIGURE 1. A schematic representation of the device used for solving an instance with 3 numbers. On each arc we have depicted its length. The loops have the lengths a_i $(1 \leq i \leq 3)$ and the arc connecting the computational node with the destination node has length k. The light is initially sent to from the start node to the computational node.

4.4. **How the system works.** In the graph depicted in Figure 1 the light enters the computational node. It is divided into $card(A) + 1$ subrays of smaller intensity. One of these rays arrives into the destination node and the rest of them enter the loops, arriving again to the computational node at moments a_i $(1 \leq i \leq card(A))$. Each of these subrays will be divided into $card(A) + 1$ subrays, one going to the destination node and the rest going back to the 1^{st} node, arriving there at moments $a_i + a_j$ $(1 \leq i \leq card(A), 1 \leq j \leq card(A))$.

Let us assume that we have a problem with 3 numbers, as shown in Figure 1. In this case, the moments when the rays enter the *computational node* are:

{
0,
$a_1, a_2, a_3,$
$2 * a_1, a_1 + a_2, a_1 + a_3, a_2 + a_1, 2 * a_2, a_2 + a_3, a_3 + a_1, a_3 + a_2, 2 * a_3$
...
}.

We have given these numbers as sets because some of them can be equal.

In the destination node we have more rays arriving at different moments. There can be two cases:

- If there is a ray arriving at moment $B + k$, this means that there is a combination of numbers from A which can be summed up to the B value.
- If there is a no ray arriving at moment $B + k$, it means that there is no way to combine numbers from A in order to get sum B.

If there is more than one ray arriving at moment $B + k$ to the destination it simply means that there are multiple multisets whose elements have the same sum. This is not a problem for us because we want to solve the YES/NO decision problem (see section 3). At this moment we are not interested in finding the subset generating the solution.

Because we work with continuous signal we cannot expect to have discrete output at the destination node. This means that arrival of the rays is notified by fluctuations in the intensity of the light. These fluctuations will be transformed by the photodiode in fluctuations of electric power which will subsequently be easily processed.

4.5. **Physical implementation of the system.** In order to implement the proposed device we need the following components:

- A source of light (laser),
- Several beam-splitters for dividing a light ray into multiple subrays. A standard beam-splitter is designed using a half-silvered mirror. In order to divide a ray into k subrays we need $k - 1$ beam-splitters,

- A high speed photodiode for converting light rays into electrical power. The photodiode is placed in the destination node. The results can be stored or processed later. Also they can be processed live by using a reference beam which is delayed with a B amount of time,
- A set of optical fiber cables whose lengths are equals to the numbers in the given set A. These cables are used for connecting nodes and for delaying the signals within the nodes.

5. **Analysis of the proposed device.** This section describes some of the problems of the proposed device and suggests some ways to deal with them. In section 5.1 we compute the complexity of building the device and the complexity of computing the solution. Section 5.2 investigates the precision of solution representation. Section 5.3 shows how to handle the exponential decrease of power. In section 5.4 we give an approximation for the size of the numbers that can appear in set A. Several problems that might be encountered during the physical implementation are discussed in section 5.5. Section 6 shows how to improve the device by reducing the speed of light.

5.1. **Complexity.** The time required to build the device has $O(n * B)$ complexity. We assume that all the cables are shorter than B, otherwise they cannot participate in the final solution.

Because the ray encoding the solution takes B units of time to reach the destination node we can say that the complexity is $O(B)$.

The intensity of the signal decreases exponentially with the number of nodes. That is why the power required is exponential with the cardinal of the set A.

5.2. **Precision.** Another problem is that we cannot measure the moment $B + k$ exactly. We can only make this measurement only with a given precision which depends on the tools involved in the experiments. More precisely, it depends on the response time of the photodiode.

The response time of the best photodiode available on the market is within the range of picoseconds (10^{-12} seconds). This means that we should not have two signals that arrive at 2 consecutive moments with a difference smaller than 10^{-12} seconds. We cannot distinguish them if they arrive within an interval smaller than 10^{-12}s. In our case, it simply means that if a signal arrives to the destination within the interval $[B + k - 10^{12}, B + k + 10^{12}]$, we cannot be perfectly sure that we have a correct subset or another one, which does not have the property in question.

We know that the speed of light in vacuum is $3 \cdot 10^8 m/s$. Based on that piece of information we can easily compute the minimal cable length that should be traversed by the ray in order for the latter to be delayed with 10^{-12} seconds. This is obviously 0.0003 meters.

This value is the minimal delay that should be introduced by an arc in order to ensure that the difference between the moments when two consecutive signals arrive at the destination node is greater than or equal to the measurable unit of 10^{-12} seconds. This will also ensure that we will be able to correctly identify whether the signal has arrived to the destination node at a moment equal to the sum of the delays introduced by each arc.

Please note that all the lengths must be integer multiples of 0.0003. We cannot accept cables whose lengths can be written as $p * 0.0003 + q$, where p is an integer and q is a positive real number lower than 0.0003. The reason is that by combining this kind of numbers we can have a signal within the above mentioned interval but that signal does not encode a subset whose sum is the expected one.

Once we have the length for that minimal delay, is quite easy to compute the lengths of the other cables that are used in order to induce a certain delay.

Please note that the maximal number of nodes can be increased when the precision of our measurement instruments (photodiode) is increased.

5.3. **Energy consumption.** Beam splitters are used in our approach for dividing a ray into more subrays. Because of this, the intensity of the signal decreases. In the worst case, we have an exponential decrease of the intensity. For instance, having a set A with n elements each signal is divided (within the two computational nodes) into $n + 1$ signals. Roughly speaking, the intensity of the signal will decrease $(n + 1)^{[B/min(A)]+1}$ times, where $min(A)$ represents the smallest value in the set A.

This means that we either have to be capable of detecting very small fluctuations in the intensity of the signal at the destination node or of using exponential quantities of energy. In order to read small fluctuations we will use a photomultiplier [7], which is an extremely sensitive detector of light in the ultraviolet, visible and near infrared range. This detector multiplies the signal produced by incident light by as much as 10^8, value which permits the detection of every single photon.

If we cannot read small amounts of energy we have to use large amounts of energy as the input. Unfortunately, by increasing the size of the instances we need exponentially more energy. Please note that this difficulty is not specific to our system only. Other major unconventional computation paradigms (such as DNA computing [1, 2, 5]), attempting to solve the NP-complete problems, share the same fate. For instance, a quantity of DNA equal to the mass of the Earth is required in order to solve Hamiltonian Path Problem instances of 200 cities using DNA computers [11].

5.4. **Problem size.** We are also interested in finding the size of the instances that can be solved by our device. Regarding the cardinal of A we cannot make too many approximations because it actually depends on the available power and the sensitivity of the measurement tools.

However, having a limited length (say 3 kilometers) for each cable, we can compute the maximal value for the numbers that can appear in A. We know that each number is lower than or equal to B. This is why we want to see how high B can be.

Without reducing generality we may assume that all the numbers are positive integers. We know that the shortest possible delay is 0.0003 meters (see section 5.2). Having a cable of 3 kilometers we may encode numbers lower than 10^7.

Longer cables may also be available. Take for instance the optical cables linking the cities in a given country. We may easily find cables having a length of 300 km. In this case we may work with numbers lower than 10^9. This is very close to the largest integer value represented over 32 bits.

5.5. **Technical challenges.** There are many technical challenges that must be solved when implementing the proposed device. Some of them are:

- Cutting the optic fibers to an exact length with high precision. Failure in accomplishing this task can lead to errors in detecting whether there has been a fluctuation in the intensity at moment $B + k$.
- Finding a high precision photodiode. This is an essential step for measuring the moment $B + k$ (see section 5.2).

6. **Improving the device by reducing the speed of the signal.** The speed of the light in the optic fibers is an important parameter in our device. The problem is that light is too fast for our measurement tools. We can either increase the precision of our measurement tools or decrease the speed of light.

It is well known that the speed of light traversing a cable is significantly smaller than the speed of light in void. Commercially available cables have a limit of the speed at 60% from the original speed of light. This means that we can obtain the same delay by using a shorter cable.

By reducing the speed of light by 7 orders of magnitude, we can reduce the size of the involved cables by a similar order (assuming that the precision of the measurement tools is still the same). This will help us solve larger instances of the problem.

7. **Advantages and weaknesses.** Several important advantages can be obtained by using the proposed device/algorithm:

- The algorithm is implemented natively on hardware. There is no distinction between the hardware and the software. This could lead to better performances compared with the standard, general-purposes, devices.
- The device is massively parallel. No simulations of the parallelism are made.
- The unbounded subset sum is a NP-complete problem [8]. There is no way to perfectly solve it unless all the possible combinations of numbers from the given set are generated. This is what our device does: it generates all possible solutions and chooses the good one.

There are other approaches for solving the unbounded subset sum problem. These approaches fall into 2 categories:

- Heuristics. These algorithms cannot guarantee the optimal solution even for very small instances. Our hybrid device-algorithm can guarantee solution for small instances.
- Brute-force. These algorithms generate all the possible combinations and output the correct one. Our device being natively implemented into the hardware can solve larger instances than other algorithms using a general hardware (such a standard computer).

Our device has some disadvantages too:

- It is hard to construct a scalable device which can easily accomodate larger instances of the problem,
- Solving larger instances requires large quantities of energy. This aspect has been discussed in section 5.3.

8. **Conclusions and further work.** The way in which light can be used for performing useful computations has been suggested in this paper. The techniques are based on the massive parallelism of the light ray.

The way in which a light-based device can be used for solving the unbounded subset-sum problem has been shown. Using nowadays technology, we can build a light-based device which can solve small and medium-sized instances.

Further work directions will be focused on:

- Automating the entire process from building the devices to the interpretation of the results. This is an important step for making the device practical,
- Using a reference beam for computing the moment when the solution arrives into the destination node. The reference beam will be delayed by exactly B units of time. In this case we will not need the photodiode anymore,
- Our device cannot find the set of numbers representing the solution. It can only say if there is such a subset or not. If there are multiple paths, we cannot distinguish them. However, the unbounded subset sum YES/NO decision problem is still an NP-complete problem [8]. We are currently carrying out a research into finding a way to store the order of the arcs so that we can easily reconstruct the path,
- Finding other ways to introduce delays in the system. The current solution requires cables that are too long and too expensive,
- Using other types of signals instead of light. Possible candidates are electric power and sound,
- Finding other ways to implement the system of marking the signals which pass through a particular node. This will be very useful because the currently suggested system, based on delays, is too time consuming. The other properties of light which could be used for this purpose are: amplitude, wavelength, etc.

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